- $(\{\gamma n^2\})_{n=1}^{\infty}$ is equidistributed if $\gamma \notin \mathbb{Q}$.

- $\left(\{P(n)\}\right)_{n=1}^{\infty}$ is equidistributed where $P(x) = c_N x^N + \cdots + c_0$ and one of $c_1, \cdots, c_N$ is irrational.

Lemma Let $f : \mathbb{N} \to \mathbb{R}$ be a function.

Then $\exists\, C > 0$ s.t. $\forall N \in \mathbb{N}$, $\forall H \leq N$,

$$\left|\sum_{n=1}^{N} e^{2\pi i f(n)}\right|^2 \leq C \frac{N}{H} \sum_{k=0}^{H-1} \left|\sum_{n=1}^{N-k} e^{2\pi i (f(n+k) - f(n))}\right|$$

Pf of Lemma:

Let $a_n = \begin{cases} e^{2\pi i f(n)}, & 1 \leq n \leq N, \\ 0, & \text{otherwise}. \end{cases}$

$$S_N = \sum_{n=1}^{N} a_n = \sum_{n=-\infty}^{\infty} a_n$$

We want $|S_N|^2 \leq C \frac{N}{H} \sum_{k=0}^{H-1} \left|\sum_{n=1}^{N-k} e^{2\pi i (f(n+k) - f(n))}\right|$

$$= C \frac{N}{H} \sum_{k=0}^{H-1} \left|\sum_{n=1}^{N-k} a_{n+k} \overline{a_n}\right|$$

i.e. $$|HS_N|^2 \leq CNH \sum_{k=0}^{H-1} \left| \sum_{n=1}^{N-k} a_{n+k} \overline{a_n} \right| .$$

$$\begin{aligned} HS_N &= H \sum_{n=1}^{N} a_n \\ &= \sum_{k=1}^{H} \sum_{n=1}^{N} a_n \\ &= \sum_{k=1}^{H} \sum_{n=1-H}^{N-1} a_{n+k} \\ &= \sum_{n=1-H}^{N-1} \sum_{k=1}^{H} a_{n+k} \end{aligned}$$

$$\begin{aligned} |HS_N|^2 &\leq \left(\sum_{n=1-H}^{N-1} 1 \right) \cdot \left(\sum_{n=1-H}^{N-1} \left| \sum_{k=1}^{H} a_{n+k} \right|^2 \right) \quad \text{Cauchy-Schwarz.} \\ &= (N+H-1) \left(\sum_{n=1-H}^{N-1} \left| \sum_{k=1}^{H} a_{n+k} \right|^2 \right) \\ &\leq 2N \left(\sum_{n=1-H}^{N-1} \left| \sum_{k=1}^{H} a_{n+k} \right|^2 \right) \\ &\leq 2N \sum_{n=-\infty}^{\infty} \left| \sum_{k=1}^{H} a_{n+k} \right|^2 \end{aligned}$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \left| \sum_{k=1}^{H} a_{n+k} \right|^2 &= \sum_{n=-\infty}^{\infty} \sum_{i=1}^{H} \sum_{j=1}^{H} a_{n+i} \overline{a_{n+j}} \\ &= \sum_{i=1}^{H} \sum_{j=1}^{H} \sum_{n=-\infty}^{\infty} a_{n+i} \overline{a_{n+j}} \\ &= \underbrace{\sum_{1 \leq i < j \leq H} \sum_{n=-\infty}^{\infty} a_{n+i} \overline{a_{n+j}}}_{(\mathrm{I})} + \underbrace{\sum_{1 \leq j < i \leq H} \cdots}_{(\mathrm{II})} + \underbrace{\sum_{k=1}^{H} \sum_{n=-\infty}^{\infty} |a_{n+k}|^2}_{(\mathrm{III})} \end{aligned}$$

$$(\mathrm{I}) = \sum_{1\le i<j\le H} \sum_{n=-\infty}^{\infty} a_{n+i}\overline{a_{n+j}} = \sum_{i=1}^{H-1}\sum_{j=i+1}^{H}\sum_{n=-\infty}^{\infty} a_{n+i}\overline{a_{n+j}}$$

$$= \sum_{i=1}^{H-1}\sum_{j=i+1}^{H}\sum_{n=-\infty}^{\infty} a_n\overline{a_{n+j-i}}$$

$$= \sum_{i=1}^{H-1}\sum_{k=1}^{H-i}\sum_{n=-\infty}^{\infty} a_n\overline{a_{n+k}} \qquad (k=j-i)$$

$$= \sum_{k=1}^{H-1}\sum_{i=1}^{H-k}\sum_{n=-\infty}^{\infty} a_n\overline{a_{n+k}}$$

$$= \sum_{k=1}^{H-1}(H-k)\sum_{n=-\infty}^{\infty} a_n\overline{a_{n+k}}$$

$$b_n = \overline{a_n}$$

$$(\mathrm{II}) = \sum_{k=1}^{H-1}(H-k)\sum_{n=-\infty}^{\infty} b_n\overline{b_{n+k}}$$

$$= \sum_{k=1}^{H-1}(H-k)\sum_{n=-\infty}^{\infty} \overline{a_n}\,a_{n+k}$$

$$(\mathrm{III}) = \sum_{k=1}^{H}\sum_{n=-\infty}^{\infty}|a_{n+k}|^2 = \sum_{k=1}^{H}\sum_{n=-\infty}^{\infty}|a_n|^2 = H\sum_{n=-\infty}^{\infty}|a_n|^2$$

$$(\mathrm{I})+(\mathrm{II})+(\mathrm{III}) = H\sum_{n=-\infty}^{\infty}|a_n|^2 + \sum_{k=1}^{H-1}(H-k)\,2\mathrm{Re}\left(\sum_{n=-\infty}^{\infty}\overline{a_n}\,a_{n+k}\right)$$

$$\le 2H\left(\sum_{n=-\infty}^{\infty}|a_n|^2 + \sum_{k=1}^{H-1}\left|\sum_{n=-\infty}^{\infty}\overline{a_n}\,a_{n+k}\right|\right)$$

$$= 2H\left(\sum_{k=0}^{H-1}\left|\sum_{n=-\infty}^{\infty}\overline{a_n}\,a_{n+k}\right|\right)$$

$$= 2H\left(\sum_{k=0}^{H-1}\left|\sum_{n=1}^{N-k}\overline{a_n}\,a_{n+k}\right|\right)$$

$\square$

Theorem : $(\{\gamma n^2\})_{n=1}^{\infty}$ is equidist iff $\gamma \notin \mathbb{Q}$.

Pf: Fix $k \in \mathbb{Z} \setminus \{0\}$. Fix $\varepsilon > 0$.

$$\left|\sum_{n=1}^{N} e^{2\pi i k \gamma n^2}\right|^2 \le C \frac{N}{H} \sum_{h=0}^{H-1} \left|\sum_{n=1}^{N-h} e^{2\pi i k \gamma [(n+h)^2 - n^2]}\right|$$

$$= C \frac{N}{H} \sum_{h=0}^{H-1} \left|\sum_{n=1}^{N-h} e^{2\pi i k \gamma (2nh + h^2)}\right|$$

$$= C \frac{N}{H} \sum_{h=1}^{H-1} \left|\sum_{n=1}^{N-h} e^{4\pi i k \gamma n h}\right| + C \cdot \frac{N^2}{H}$$

$$\le C \frac{N}{H} \sum_{h=1}^{H-1} \left(\left|\sum_{n=1}^{N} e^{4\pi i k \gamma n h}\right| + h \right) + C \cdot \frac{N^2}{H}$$

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \gamma n^2}\right|^2 \le \frac{C}{H}\left(1 + \frac{H-1}{2N} + \frac{1}{N}\sum_{h=1}^{H-1}\left|\sum_{n=1}^{N} e^{4\pi i k \gamma n h}\right|\right)$$

(Choose H s.t. $\frac{C}{H} < \varepsilon$)

$$< \varepsilon \left(1 + \frac{H-1}{2N} + \frac{1}{N}\sum_{h=1}^{H-1}\left|\sum_{n=1}^{N} e^{4\pi i k \gamma n h}\right|\right)$$

$$< \varepsilon \left(1 + \frac{H-1}{2N} + \frac{1}{N}(H-1)M\right)$$

(Choose N large)

$$< 2\varepsilon$$

$\square$

Lemma: If $(\{x_{n+h} - x_n\})_{n=1}^{\infty}$ is equidist, $\forall h \in \mathbb{Z}^+$, then $(\{x_n\})_{n=1}^{\infty}$ is equidistist.

Theorem: $(\{P(n)\})_{n=1}^{\infty}$ is equidist where $P(x) = c_N x^N + \cdots + c_0$ and one of $c_1, \cdots, c_N$ is irrational.

Pf: We prove by induction on the highest degree whose coefficient is irrational.

Let $S(m)$ be the statement

If $P(x) = c_N x^N + \cdots + c_0$ with $c_m \notin \mathbb{Q}$ and $c_k \in \mathbb{Q}$, $\forall k > m$, then $(\{P(n)\})_{n=1}^{\infty}$ is equidist.

- $S(1)$ is true
- $S(m) \Rightarrow S(m+1)$

Let $P(x) = \underbrace{c_N x^N + \cdots + c_{m+2}x^{m+2}}_{Q(x)} + c_{m+1}x^{m+1} + \underbrace{c_m x^m + \cdots + c_0}_{R(x)}$

$Q(n+h) - Q(n)$ has rational coefficient

$c_{m+1}(n+h)^{m+1} - c_{m+1}n^{m+1} = c_{m+1}(m+1)hx^m + (\text{degree} \le m-1)$

$R(n+h) - R(n)$ has degree $\le m-1$.

So $(\{P(n+h) - P(n)\})_{n=1}^{\infty}$ is equidist $\forall h \in \mathbb{Z}^+$ by our induction assumption

By Lemma, $(\{P(n)\})_{n=1}^{\infty}$ is equidist. #

. $S(1)$ is true.

Let $P(x) = Q(x) + \gamma x + c_0$ where $Q(x)$ has rational effieients

Fix $k \in \mathbb{Z} \setminus \{0\}$. We want $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k P(n)} \to 0$.

Take $L \in \mathbb{N}$ s.t. $Lc_2, \cdots, Lc_N \in \mathbb{Z}$.

Write $N = aL + b$, $0 \le b < L$.

$$\left|\sum_{n=1}^{N} e^{2\pi i k P(n)}\right| \le \left|\sum_{n=1}^{aL} e^{2\pi i k P(n)}\right| + L$$

$$\sum_{n=1}^{aL} e^{2\pi i k P(n)} = \sum_{c=0}^{a-1} \sum_{d=1}^{L} e^{2\pi i k P(cL+d)}$$

$$= e^{2\pi i k c_0} \sum_{d=1}^{L} \sum_{c=0}^{a-1} e^{2\pi i k Q(d)} \cdot e^{2\pi i k \gamma(cL+d)}$$

$$\left|\sum_{n=1}^{aL} e^{2\pi i k P(n)}\right| \le \sum_{d=1}^{L} \sum_{c=0}^{a-1} e^{2\pi i k \gamma c L}$$

$$< LM$$

$\square$